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On the Invariance of the Factors of Composition of a Substitution-group.

By James Pierpont, *New Haven, Conn.*

The theorem* which asserts the invariance of the factors of composition of a group is of great importance in the theory of algebraic equations. It may be stated thus:

In whatever manner we may decompose a group into a series of composition

$$G, \; G_1, \; G_2, \; G_3, \; \ldots , \; G_\mu = 1,$$

the factors of composition

$$\alpha_1, \; \alpha_2, \; \alpha_3, \; \ldots , \; \alpha_\mu$$

are in every case the same, aside from their order. (A)

Jordan's lengthy demonstration of this theorem has been greatly simplified by Netto† in his "Treatise on the Theory of Substitutions"; but by employing the notion of isomorphism the proof may be made still more simple.‡

The conception of isomorphism may be arrived at as follows: Let the equation $F(x) = 0$ have for a certain region of rationality the group G. Let ϕ_1 be any rational function of the roots of $F(x) = 0$ belonging to a subgroup G_1 of G, which for G takes on the values

$$\phi_1, \; \phi_2, \; \phi_3, \; \ldots , \; \phi_m .$$

Any substitution of G operating on this series will, in general, cause the ϕ's to permute among themselves, and thus give rise to a group of substitutions Γ

* C. Jordan, Traité des Substitutions et des Équations Algébriques. Paris, 1870, p. 41–48.

† E. Netto, Substitutionentheorie und ihre Anwendung auf die Algebra. Leipzig, 1882, p. 87–90.

‡ Cf. J. König, Ueber rationale Functionen, etc. Mathematische Annalen, vol. XIV, p. 212–30.

O. Hölder, Zurückführung einer beliebigen algebraischen Gleichung auf eine Kette von Gleichungen. Math. Ann., vol. XXXIV, p. 26–56.

which, having the same properties as those of G, are said to be isomorphic to G. To a substitution of G will correspond a substitution of Γ, while to every substitution of Γ will correspond one and the same number of substitutions of G, namely, the number of substitutions of G corresponding to the identical substitution 1 of Γ.

The proof of (A) depends upon the following lemma:

$$\left.\begin{array}{l}\text{Let the group } G \text{ of order } r \text{ have two maximal invariant subgroups } H \\ \text{(order } h\text{, index } \alpha\text{) and } K \text{ (order } k\text{, index } \beta\text{). The group of substitutions} \\ C \text{ (order } c\text{) common to } H \text{ and } K \text{ is a maximal invariant subgroup for } H \\ \text{and } K. \text{ The index } \sigma \text{ of } C \text{ under } H \text{ is equal to } \beta\text{, while its index } \tau \text{ under} \\ K \text{ is equal to } \alpha.\end{array}\right\} \text{(B)}$$

We proceed then to prove (B).

In the first place we notice that C is invariant for H and K. For $S = H^{-1}CH$ is a substitution of H or K according as we regard C as in H or K. Thus S is common to H and K, and hence is in C, so that $H^{-1}CH = C$.

Let now ϕ be a rational function of the roots of $F(x) = 0$ belonging to C, and let $\Gamma = (\gamma_1, \gamma_2 \ldots .)$ be the group of substitution in the ϕ's corresponding to G; while $\Delta = (\delta_1, \delta_2 \ldots .)$ and $\Theta = (\vartheta_1, \vartheta_2 \ldots .)$ may correspond to H and K resp. Then Δ and Θ are maximal invariant subgroups of Γ of orders σ and τ resp. The order of Γ is $\rho = \dfrac{r}{c}$.

The groups Δ and Θ have only the identical substitution in common; for if σ be common to them, then the corresponding substitutions $s_1, s_2 \ldots .$ of G are common to H and K and are hence in C. But to the substitutions of C correspond the identical in Γ.

The substitutions δ, ϑ are commutative. For the substitution $\delta^{-1}\vartheta^{-1}\delta\vartheta$ is equal to the identical substitution, since it is common to Δ and Θ, whence

$$\delta\vartheta = \vartheta\delta. \tag{a}$$

The substitutions of Γ are all of the form

$$\gamma = \delta\vartheta. \tag{b}$$

For these substitutions form a subgroup of Γ which is invariant for Γ, since Δ and Θ are. As this group contains the maximal subgroups Δ and Θ, it is identical with Γ. Further, from $\delta\vartheta = \delta_1\vartheta_1$ follows $\delta_1^{-1}\delta = \vartheta_1\vartheta^{-1}$, which, being a substitution common to Δ and Θ, is the identical substitution; whence $\delta = \delta_1$ and $\vartheta = \vartheta_1$. Thus $\rho = \sigma\tau$.

Let $\Lambda = (\lambda_1, \lambda_2 \ldots .)$ be an invariant subgroup of Δ; then by (a) it is also invariant for Θ. The substitution $\pi_a = \lambda_{a_1} \vartheta_{a_2}$ form a subgroup $\Pi = (\pi_1, \pi_2 \ldots .)$ of Γ by virtue of (a), which is invariant for Γ, since Λ and Δ are. To Π will correspond the invariant subgroup P of G. As Π contains Θ, P will contain K, which being a maximal invariant subgroup of G requires that $\Lambda = 1$, so that C is a maximal invariant subgroup for Δ or Θ.

Finally since $r = c\sigma\tau = c\alpha\sigma = c\beta\tau$; $\alpha = \tau$, $\beta = \sigma$.

The demonstration of (A) now follows easily as by Netto.

Let $$G,\ G_1,\ G_2,\ G_3 \ldots . \tag{1}$$
and $$G,\ H_1,\ H_2,\ H_3 \ldots . \tag{2}$$
be two series of composition for G, having the factors of composition
$$\alpha_1,\ \alpha_2,\ \alpha_3 \ldots . \tag{α}$$
and $$\beta_1,\ \beta_2,\ \beta_3 \ldots . \tag{β}$$
Then if Γ be the substitution of G common to G_1 and H_1, the series
$$G,\ G_1, \Gamma \ldots . \tag{3}$$
$$G,\ H_1, \Gamma \ldots . \tag{4}$$
having by virtue of (B) the same factors, the demonstration of the identity of (α), (β) is reduced to proving that the factors of (1) are the same as those of (3); also that the factors of (2) and (4) are alike. But while (1), (2) have only the group G in common, (1) and (3) have e. g. two groups, G and G_1 in common. Applying the same reasoning to (1), (3), we obtain two new series having three groups in common, and so on. Finally, we need to show that if one mode of decomposition of a certain group L be
$$L,\ M,\ 1$$
while another is $$L,\ R \ldots .,$$
then these two series have the same factors. But this is evident from (B).

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